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AUTOMATIC CONTINUITY AND A PROBLEM OF KAPLANSKY

- by -

JOHN A. AUSINK

Submitted in partial fulfilment of the requirements for the Degree of Master of Science at the University of Oxford

November 1977

Merton College

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CONTENTS

INTRODUCTION

Automatic continuity is the study of restrictions that can be imposed upon the domain and/or range of an operator that will guarantee the continuity of the operator. Theorems of this nature are interesting by themselves, but they sometimes have surprising applications as well - the use of a continuity theorem in Johnson's proof [10] that all complete algebra norms in a semisimple Banach algebra are equivalent being just one example.

There is another historical link between the study of norms and automatic continuity. Whether or not there is an incomplete algebra norm for $C(X,\mathbb{C})$, the algebra of continuous, complex-valued functions on a compact Hausdorff space X, is a question that was first raised by Kaplansky [14] in 1949. Badé and Curtis [2] showed that such a norm would exist if and only if there was a discontinuous homomorphism from $C(X,\mathbb{C})$ to a Banach algebra, but despite the interest generated by their work, the question remained unresolved until very recently. It was finally announced in 1977 [7] that H.G.Dales and J.Esterle had, independently, succeeded in constructing discontinous homomorphisms from $C(X,\mathbb{C})$. Both constructions require the assumption of the continuum hypothesis.

The purpose of this dissertation is to provide an introduction to the subject of automatic continuity, with emphasis on this problem of Kaplansky, the work of Badé and Curtis, and a discussion of Dales'

construction.

In Chapter I, some of the "classical" theorems for the continuity of homomorphisms and positive linear functionals are proved to show the types of restrictions that can be involved in automatic continuity arguments. Most of these results are found in [24].

Chapter II begins with an outline of basic representation theory, directed toward the proof of Johnson's theorem on the uniqueness of the complete algebra norm in semi simple Banach algebras. This is used as motivation for consideration of the problem of Kaplansky.

The simplification of the problem by Badé and Curtis is the subject of Chapter III. Extensions and generalizations of their results are also considered.

Chapter IV is an attempt to outline Dales' construction of a discontinuous homomorphism from $C(X,\underset{\sim}{C})$. Even Dales felt uncomfortable summarizing his paper, which is very complex, so I have contented myself with highlights and the use of the simplification of Badé and Curtis.

Throughout, a familiarity with basic concepts of Banach algebra theory and functional analysis are assumed. For completeness, however, some elementary definitions have been included, and I have tried to indicate references for results that are used without proof. Because Dales' construction and some of the theorems that lead to it embrace so many concepts, I have assumed more in the last section of Chapter III and in Chapter IV.

Specifically, knowledge of terminology of abstract algebra is necessary along with properties of the Stone-Čech compactification. Again, I have tried to include references where appropriate.

CHAPTER 1

ORIGINS AND RESULTS FOR BANACH *-ALGEBRAS

The first theorems that could be collected under the heading "automatic continuity" dealt with properties of homomorphisms from one Banach algebra into another. Non-zero homomorphisms which have as range the Banach algebra $\underset{\sim}{C}$ of complex numbers are called complex homomorphisms, and results involving them are the easiest of any interest.

§1 Homomorphisms

Any Banach algebra A can be isometrically embedded in a Banach algebra A_1 with identity, and A_1 is called the algebra with identity adjoined. Recall that if x is an element of a Banach algebra A with identity e such that $\|x-e\| < 1$, then x is invertible.

1.1 Lemma If ϕ is a complex homomorphism on a Banach algebra A with identity e, then $\phi(e) = 1$ and $\phi(x) \neq 0$ for every invertible x.

Proof. ϕ is non-zero, so $\phi(y) \neq 0$ for some y in A. Since $\phi(y) = \phi(ye) = \phi(y)\phi(e)$, $\phi(e) = 1$. If x is invertible, $1 = \phi(e) = \phi(x^{-1}x) = \phi(x^{-1})\phi(x)$ and so $\phi(x) \neq 0$.

1.2 Theorem If ϕ is a complex homomorphism on a Banach algebra A with identity e, then ϕ is continuous.

Proof We show that $\emptyset$ is bounded. Suppose there exists z in A such that $\emptyset(z) > \|z\|$.

$\emptyset(z) \neq 0$, so we may write $x = {}^{z}/\emptyset(z)$, and $\emptyset(x) = 1$. Since $\|x\| < 1$, (e-x) is invertible. Thus,

$\emptyset(e-x) = \emptyset(e) - \emptyset(x) = 1 - \emptyset(x) \neq 0$ and $\emptyset(x) \neq 1$.

This contradiction shows $\emptyset(z) \leq \|z\|$ for all z in A as required.

Although very simple, Theorem 1.2 is remarkable because it relates the algebraic notion of homomorphism to the analytic one of continuity. Bachman [1:334] remarks, "this is something like saying $\emptyset$ is a homomorphism, therefore it is green". More importantly, it is "the seed ... from which automatic continuity grew". [24:1].

If A is a commutative Banach algebra with identity, the set of all complex homomorphisms on A is denoted by Δ_A. The radical of A, rad (A), is given by $\text{rad}(A) = \cap\{\text{Ker}\emptyset : \emptyset \varepsilon \Delta_A\}$ and A is said to be semisimple if $\text{rad}(A) = \{0\}$.

1.3 Theorem [20:269] If $\psi: A \to B$ is a homomorphism and A and B are commutative, semisimple Banach algebras with identity, then ψ is continuous.

Proof. Suppose $x_n \to x$ in A and $\psi(x_n) \to y$ in B. Let Δ_A and Δ_B denote the sets of complex homomorphisms in A and B. Fix $h \varepsilon \Delta_B$ and let $\emptyset = h \circ \psi$. Then $\emptyset \varepsilon \Delta_A$, and by Theorem 1.2 h and $\emptyset$ are continuous. Hence

$$h(y) = \lim h(\psi(x_n)) = \lim \emptyset(x_n) = \emptyset(x) = h(\psi(x))$$

for every $h \varepsilon \Delta_B$. Thus, $y - \psi(x)$ is in the radical of B, and so $y = \psi(x)$. By the Closed Graph Theorem, ψ is continuous.

1.4 Definition An involution on a Banach algebra A is a map $*:A \to A$ such that for all $x \in A$, $\lambda \in \mathbb{C}$,

1) $(x + y)^* = x^* + y^*$
2) $(\lambda x)^* = \bar{\lambda} x^*$
3) $(xy)^* = y^* x^*$
4) $x^{**} = x$

A Banach algebra with involution is called a Banach *-algebra. If in addition

5) $\| x^* x \| = \| x \|^2$

it is called a C*-algebra.

1.5 Theorem [20:276] If the Banach algebra A with identity is commutative and semisimple, then every involution on A is continuous.

Proof Let h be a complex homomorphism on A and define $\emptyset(x) = \overline{h(x^*)}$ By the properties of an involution, $\emptyset$ is a complex homomorphism and is therefore continuous. Suppose $x_n \to x$ and $x_n^* \to y$. Then $\overline{h(x^*)} = \emptyset(x) = \lim \emptyset(x_n) = \lim \overline{h(x_n^*)} = \overline{h(y)}$. Since A is semisimple, $y = x^*$ and * is continuous by the Closed Graph Theorem.

The Closed Graph Theorem is needed frequently in theorems of this nature, and its use has prompted the following definiton.

1.6 Definition If S is a linear operator from a Banach space X into a Banach space Y, the separating space of S, denoted $\mathfrak{S}(S)$, is given by

$$\mathfrak{S}(S) = \{y \in Y: \text{there is a sequence } \{x_n\} \text{ in } x \text{ with } x_n \to 0 \text{ and } Sx_n \to y\}.$$

It is easy to see that $\mathfrak{S}(S) = \{0\}$ if and only if S is continuous. We will have occasion to use the separating space in Chapters II and III.

§2 Positive Linear Functionals

1.7 Definition A linear functional f on a Banach *-algebra A is positive if $f(x^*x) \geq 0$ for all x in A.

Several interesting theorems concerning the continuity of positive linear functionals can be proved, but first we recall three concepts.

1.8 Definition If A is a Banach algebra with identity, the spectrum of x, denoted sp(x) is $sp(x) = \{\lambda \varepsilon C: (\ e-x) \text{ is not invertible in } A\}$. If A does not have an identity, sp(x) is the spectrum of x considered as an element of A_1, the algebra with identity adjoined.

1.9 Definition The number $\nu(x) = \sup\{|\lambda|:\lambda \varepsilon sp(x)\}$ is the spectral radius of x, and it can be shown [20:235] that $\nu(x) = \inf \|x^n\|^{1/n} = \lim \|x^n\|^{1/n}$.

1.10 Functional Calculus Theorem [16:12] Let A be a Banach algebra and $x \varepsilon A$. If f is a complex valued function defined and analytic on a neighbourhood of sp(x)(and satisfying $f(0) = 0$ if A has no identity) then there exists an element f(x) in A such that $sp(f(x)) = f(sp(x))$.

The following lemma was first proved by Ford [8]. Its importance rests in the fact that before its proof, similar results could be obtained only by assuming that the involution was continuous. This proof is due to Sinclair [27:24] and is a nice application of the functional calculus.

1.11 Lemma Let A be a Banach *-algebra. Let $a = a^*$ be an element in A with $sp(a) \cap [1,\infty) = \emptyset$. Then there is a unique $x = x^*$ in A satisfying

$sp(x) \subseteq \{z \in \mathbb{C} : \mathrm{Re}\, z < 1\}$ and $2x - x^2 = a$.

Proof Let A_1 be A with identity adjoined. Let $f(z) = 1 - (1 - z)^{\frac{1}{2}}$ be analytic in the domain $C \setminus [1,\infty)$ and use Theorem 1.10 to define $x = f(a)$. Then $(1 - x)^2 = 1 - a$ and $sp(x) \subseteq \{z \in \mathbb{C} : \mathrm{Re}\, z < 1\}$. Hence x^* also satisfies

$$(1 - x^*)^2 = 1 - a \text{ and } sp(x^*) \subseteq \{z \in \mathbb{C} : \mathrm{Re}\, z < 1\} \quad [18:182]$$

Thus $x = x^*$ and $2x - x^2 = a$.

For uniqueness, suppose $y \in A$, $sp(y) \subseteq \{z \in \mathbb{C} : \mathrm{Re}\, z < 1\}$ and $(1-y)^2 = 1-a$. Since $a = 2y - y^2$, $ya = ay$ by definition of x. Thus,

$$sp(x + y) \subseteq \{z \in \mathbb{C} : \mathrm{Re}\, z < 2\} \quad [19:10]$$

Therefore, $x + y - 2$ is invertible and cannot be equal to zero. Finally,

$$(1-x)^2 = 1-a = (1-y)^2, \text{ so } 0 = (x-y)(x+y-2) \text{ and } x = y.$$

The next theorem has as a corollary our first continuity result.

1.12 Theorem [24:74] If a, b, $x = x^*$ are in a Banach $*$-algebra A and if f is a positive linear functional on A, then

i) $f(a^*b) = \overline{f(b^*a)}$

ii) $|f(a^*b)|^2 \leq f(a^*a)f(b^*b)$

iii) $|f(a^*xa)| \leq f(a^*a)\nu(x)$

iv) $|f(a^*ba)| \leq f(a^*a)\nu(b^*b)^{\frac{1}{2}}$

Proof Let $\alpha, \beta \in \mathbb{C}$. Then

$$f[(\alpha a + \beta b)^*(\alpha a + \beta b)] = |\alpha|^2 f(a^*a) + \bar{\alpha}\beta f(a^*b) + \alpha\bar{\beta} f(b^*a) + |\beta|^2 f(b^*b) \geq 0.$$

This implies that $\bar{\alpha}\beta f(a^*b) + \alpha\bar{\beta} f(b^*a)$ is real for all $\alpha, \beta \in \mathbb{C}$.

i) If $\alpha = \beta = 1$, $f(a^*b) + f(b^*a)$ is real.

If $\alpha = 1$, $\beta = i$, $if(a^*b) - if(b^*a)$ is real.

Thus,

$$f(a^*b) + f(b^*a) = \overline{f(a^*b)} + \overline{f(b^*a)}$$

$$f(a^*b) - f(b^*a) = \overline{f(b^*a)} - \overline{f(a^*b)}, \text{ and}$$

$$f(a^*b) = \overline{f(b^*a)}.$$

ii) This is proved by letting $\alpha = 1$ and $\beta = -\dfrac{f(b^*a)}{f(b^*b)}$.

iii) Assume $\nu(x) < 1$. By Lemma 1.11, there are y and z in A such that $2y-y^2 = x$ and $2z-z^2 = -x$.

Let $v = a - ya$ and $w = a - za$. Then

$$v^*v = a^*(1-y)^2a = a^*(1-x)a$$

$$w^*w = a^*(1-z)^2a = a^*(1+x)a. \text{ Thus}$$

$$f(a^*a) - f(a^*xa) = f(v^*v) \geq 0$$

$$f(a^*a) + f(a^*xa) = f(w^*w) \geq 0, \text{ and}$$

$$|f(a^*xa)| \leq f(a^*a)$$

iv) By (ii), $|f(a^*ba)|^2 = |f(a^*(ba))|^2 \leq f(a^*a)f(a^*b^*ba)$, so by (iii), $|f(a^*ba)|^2 \leq f(a^*a)^2\nu(b^*b)$

The proof of the next corollary depends upon an important result (Theorem 2.18) in Chapter II, and will be postponed until that chapter as well.

1.13 <u>Corollary</u> There is a constant m such that $|f(a^*ba)| \leq mf(a^*a)\|b\|$ for all a and b in A, and all positive linear functionals f on A. In particular, if A has an identity, then every positive linear functional is continuous.

We now consider the problem of continuity of positive linear functionals when the algebra does not have an identity. After two more definitions, a theorem due to Murphy will lead to a conditional solution of the problem.

1.14 Definition A linear functional f is said to dominate a linear functional g if f-g is positive.

1.15 Definition For a Banach*-algebra A, let

$$A^K = \left\{ \sum_{i=1}^{n} x_{i1}\, x_{i2} \ldots x_{iK} : x_{ij}\ A,\ n \geq 1 \right\}$$

1.16 Theorem [17.171] Let A be a Banach *-algebra. Let $A^2 = A$ and let every non-zero positive linear functional on A dominate a continuous non-zero positive linear functional. Then every positive linear functional on A is continuous.

Proof First, the identity

$$(\dagger)\quad 4ab = (b+a^*)^*(b+a^*) - (b-a^*)^*(b-a^*) + i(b+ia^*)^*(b+ia^*) - i(b-ia^*)^*(b-ia^*)$$

implies that because every x in A can be expressed $x = \sum_{i=1}^{n} a_i b_i$, it can also be expressed $x = \sum_{i=1}^{n} \alpha_i x_i^* x_i$ $(\alpha_i \in \mathbb{C})$.

Thus, linear functionals which agree on all elements x^*x are identical. Let F be a non-zero postive linear functional and define the family

$$S = \{G : G \neq 0,\ G \text{ continuous, linear, positive and } F \text{ dominates } G\}.$$

S is non-empty by hypothesis, and we can define a partial ordering in S by $G_1 > G_2$ if and only if G_1 dominates G_2.

Let T be a totally ordered subset of S under >. For all $y \epsilon A$, $\lim G(y^*y)$ $(G \epsilon T)$ exists because $G(y^*y) < F(y^*y)$ for all $G \epsilon T$. It is thus possible to define a functional

$\emptyset(x) = \lim G(x)$ which is positive, linear and dominated by F. For every $G \epsilon T$, for all $x \epsilon A$

$$|G(x)| = |G(\sum_{i=1}^{n} \alpha_i x_i^* x_i)| \leq \sum_{i=1}^{n} |\alpha_i| G(x_i^* x_i) \leq \sum_{i=1}^{n} |\alpha_i| F(x_i^* x_i).$$

Hence, by the uniform boundedness theorem, there exists an m such that $\| G \| \leq m$ for all $G \epsilon T$, and

$$|\emptyset(x)| \leq \lim |G(x)| \leq m \| x \|$$

Thus, $\emptyset$ is continuous, $\emptyset \epsilon S$, and $\emptyset$ is an upper bound for T. By Zorn's Lemma, S has a maximal element, G_o.

Suppose $F - G_o \neq 0$. Then by hypothesis there exists a non-zero continuous, positive linear functional G_1 such that $F - G_o - G_1$ is positive. Hence $G_o + G_1 \epsilon S$. But $G_o + G_1 > G_o$, which contradicts the maximality of G_o, so $G_1 = 0$. This implies $F - G_o = 0$ and $F = G_o$, so F is continuous.

1.17 <u>Corollary</u> [17:172] If A is a commutative Banach*-algebra such that $A^2 = A$, then every positive linear functional on A is continuous.

<u>Proof</u> Let f be a non-zero positive linear functional on A and define

$f_u(x) = f(u^*xu)$, where $u \epsilon A$.

f_u is continuous by Corollary 1.13.

Suppose that $f_u = 0$ for all u in A. Then by Theorem 1.12,

$|f(u^*x^*y)|^2 \leq f(u^*x^*xu)f(y^*y) = 0.$ This implies $f(A^3) = f(A) = 0,$

which is not true because f is non-zero. For some u, then, $f_u \neq 0$. Since $f_{\alpha u} = |\alpha|^2 f_u$, we can safely assume that $\|u^*u\| < 1$. Therefore $sp(u^*u) \subseteq (0,1)$, and by lemma 1.11, there is a unique $x = x^*$ in A such that $(1-x)^2 = 1 - u^*u$.

Now, f dominates f_u because

$$
\begin{aligned}
(f - f_u)(y^*y) &= f(y^*y - u^*y^*yu) \\
&= f(y^*(1-u^*u)y) \quad \text{using commutativity} \\
&= f(y^*(1-x^*)(1-x)y) \geq 0 \quad \text{by definition of } f.
\end{aligned}
$$

The hypotheses of Theorem 1.16 are now satisfied, and we are through.

Restrictions of a different nature also yield information about the continuity of positive linear functionals. Recall that if M is a subspace of a Banach space X, then the codimension of M in X is the dimension of the factor space X/M. We require a definition and a lemma.

1.18 Definition $A^+ = \{\Sigma a_j^* a_j : \{a_1, \ldots, a_n\}$ is a finite subset of $A\}$.

1.19 Lemma [24:77] If A^+ is closed and if f is a positive linear functional on A, then there is a constant m such that $f(x) \leq m\|x\|$ for all x in A^+.

Proof Suppose there is no such constant m. It is thus possible to choose a sequence $\{x_n\}$ in A^+ such that $f(x_n) > 2^n \|x_n\|$ for all n. Now let $y_m = \sum_{n=m}^{\infty} 2^{-n}\|x_n\|^{-1}x_n$ for $m = 1,2, \ldots$. Because A^+ is closed, y_m is in A^+ for all m. Also, $y_1 = \sum_{n=1}^{m-1} 2^{-n}\|x_n\|^{-1}x_n + y_m$.

$$\text{Thus, } f(y_1) \geq f\left(\sum_{n=1}^{m-1} 2^{-n}\| x_n\|^{-1} x_n\right) = \sum_{n=1}^{m-1} 2^{-n}\| x_n\|^{-1} f(x_n)$$

$$> \sum_{n=1}^{m-1} 2^{-n}\| x_n\|^{-1}\| x_n\| 2^n = m - 1$$

for all m, which is ridiculous. This gives the result.

1.20 Theorem [24:78] Let A be a Banach *-algebra. If A^2 is closed and of finite codimension in A, and if A^+ is closed, then each positive linear functional on A is continuous.

Proof Since A^2 is of finite codimension, and all linear functionals defined on a finite dimensional space are continuous, it suffices to show that a positive linear functional is continuous on A^2. This will be done by showing that each element x in A^2 can be expressed $x = z_1 - z_2 + i(z_3 - z_4)$ where $\| z_j \| \leq N\|x\|$ and z_j is in A^+ and applying lemma 1.18.

Let $Y = \{(x_1, x_2, x_3, x_4) : x_j \varepsilon A^+\}$ and

$Y_\alpha = \{(x_1, x_2, x_3, x_4) : \| x_j \| \leq \alpha\}$ Let

$A_\alpha{}^2 = \{a\varepsilon A^2 : \| a \| \leq \alpha\}$ and define a map $T: Y \to A^2$ by

$$T(x_1, x_2, x_3, x_4) = x_1 - x_2 + i(x_3 - x_4).$$

By identity (+) in Theorem 1.16, T(Y) is equal to A^2 and so

$A^2 = \bigcup_{n=1}^{\infty} \overline{T(Y_n)}$. By the Baire Category Theorem, there is some $\overline{T(Y_n)}$ with non-empty interior, and by a translation, 0 is in the interior of $\overline{T(Y_{2n})}$. Hence there exists a $\beta > 0$ such that $\overline{T(Y_{\alpha\beta})} \supseteq A_\alpha{}^2$ for all $\alpha > 0$. Let x be in A^2 and $\| x\| \leq 1$.

A sequence $\{y_n\}$ can be constructed in Y such that

i) $\| T(y_1 + \ldots + y_n) - x \| < 2^{-n}$

ii) $y_n \in Y_{\beta . 2^{-n+1}}$ [21:236]

If $y_n = (x_{1n}, x_{2n}, x_{3n}, x_{4n})$ let $z_j = \sum_{n=1}^{\infty} x_{jn}$ which is in A^+ because it is closed.

$$\| z_j \| = \| \sum_{n=1}^{\infty} x_{jn} \| \leq \sum_{n=1}^{\infty} \| x_{jn} \| \leq \sum_{n=1}^{\infty} (2\beta 2^{-n}) = 2\beta \quad \text{by (ii), and}$$

$$x = z_1 - z_2 + i\,(z_3 - z_4) \text{ by (i).}$$

This completes the theorem.

Corollary 1.17 and Theorem 1.20 make it possible to eliminate the presence of an identity while maintaining continuity of positive linear functionals, but the concomitant restrictions in the algebra may seem extreme. With one more definition and a lemma which we state without proof, we can obtain what is perhaps a more satisfying result.

1.21 Definition [19:3] Let Λ be a directed set. A collection $\{e(\lambda) : \lambda \varepsilon \Lambda\}$ of elements of a Banach algebra A is a bounded left approximate identity if

$e(\lambda)x \to x$ for each x in A and there exists a positive constant K such that $\| e(\lambda) \| < K$ for each $\lambda \varepsilon \Lambda$.

A bounded right approximate identity is similarly defined, and a bounded two-sided approximate identity is one which is both a left and right approximate identity.

1.22 Lemma [4:62] Let A be a Banach algebra with a bounded left approximate identity and let $z_n \varepsilon A$ with $z_n \to 0$ as $n \to \infty$. Then there exist $a, y_n \varepsilon A$ with $z_n = ay_n (n = 1, 2, \ldots)$ and $y_n \to 0$ as $n \to \infty$.

1.23 Theorem [24:79] Let A be a Banach *-algebra. If A has a bounded two-sided approximate identity, then each positive linear functional on A is continuous.

Proof Let f be a positive linear functional on A and let $\{x_n\}$ be a sequence in A with $x_n \to 0$. Then there are a, y_1, y_2 ... in A such that $x_n = ay_n$ for all n and $y_n \to 0$ by Lemma 1.22. By the right multiplication version of the lemma, there are b, z_1, z_2 in A such that $y_n = z_n b$ for all n and $z_n \to 0$.

Define $F(x) = f(axb)$. From the identity (+) of Theorem 1.16 and applying Theorem 1.13, we conclude that F is continuous.

Hence

$$f(x_n) = f(az_n b) = F(z_n) \to 0 \text{ as } n \to \infty, \text{ and f is continuous.}$$

Rickart [19:245] showed that every C*-algebra has a bounded, two-sided approximate identity.

We return now, in a rather circuitous fashion, to the consideration of homomorphisms.

CHAPTER II

UNIQUENESS OF NORMS AND A PROBLEM OF KAPLANSKY

The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a Banach algebra A are equivalent if there exist positive constants a, b such that $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ for all x in A. Since the most important theorem of this chapter (2.18) concerns the equivalence of norms on certain Banach algebras, it may appear that automatic continuity has been temporarily forgotten. It has already been noted, however, that Theorem 2.18 is required in the proof of Corollary 1.13. In addition, Theorem 2.18 itself depends critically upon a continuity theorem - indeed, some considered its proof a 'victory' of sorts for automatic continuity. Finally, it will be shown in Theorem 2.21 that the study of norms and the study homomorphisms are very closely related.

In order to deal with these results, though, it is necessary to address a topic which seems even further removed from automatic continuity.

§1 Some Representation Theory

Throughout this section, A will denote a Banach algebra over the complex field, and all linear spaces will be over the complex field as well. The definitions in this section are those of Bonsall and Duncan [4], and though they are a bit dull, they are necessary in order to achieve Theorem 2.18.

2.1 Definition A left ideal of A is a linear subspace J of A such that $AJ \subseteq J$. An element u of A is a right modular identity for a linear subspace E of A if $A(1-u) \subseteq E$. A modular left ideal is a left ideal for which there exists a right modular identity.

A left ideal J of A is proper if $J \neq A$, maximal if it is proper and not contained in any other proper left ideal, and maximal modular if it is proper, modular and not contained in any other such left ideal. Similar definitions hold for right and two-sided ideals.

2.2 Theorem [4:46] Every maximal modular left ideal in A is closed.

Recall that if L is a linear subspace of A, then the factor space A/L is a normed space under the canonical norm

$\| [x] \| = \inf \{\| y \| : y \in [x]\}$ where $[x]$ is a coset in A/L.

If L is closed, A/L is a Banach space, and if L is a closed two-sided ideal, A/L is a Banach algebra.

2.3 Definition Let M be a linear space. M is said to be a left A-module if a mapping $(a,m) \to am$ of $A \times M$ into M satisfies

1) For each fixed $a \in A$, $m \to am$ is linear on M.
2) For each fixed $m \in M$, $a \to am$ is linear on A.
3) $a_1(a_2m) = (a_1a_2)m \quad a_1, a_2 \in A, m \in M$.

The map is called module multiplication. Right modules are defined similarly. M is an A bi-module if it is both left and right and the module multiplications are related by $a(mb) = (am)b \; a,b \in A, m \in M$.

2.4 Definition A linear space M is a normed left A-module if it is a left A-module and also satisfies

$$\| am \| \leq K \| a \| \| m \| \quad a \in A,\ m \in M,\ K \text{ a positive constant.}$$

If M is complete as a normed linear space, it is a Banach left A-module and of course right Banach A-modules and Banach A-bimodules are defined similarly.

2.5 Definition Let X be a normed linear space. A representation of A on X is a homomorphism of A into L(X), the linear space of linear mappings of X into itself. If π is a representation of A on X, the corresponding left A-module is the linear space X with module multiplication

$$(*) \qquad ax = \pi(a)x$$

Conversely, given a left A-module X, the corresponding representation on X is the homomorphism π of A into L(X) given by (*).

The kernel of a representation π is given in terms of the corresponding left A-module by $\ker(\pi) = \{ a \in A : aX = \{0\}\}$.

Let L be a closed left ideal of A and let $a \to a'$ denote the canonical mapping of A onto A/L. Then A/L is a left A-module under

$$a[x] = (ay)' \quad a \in A,\ y \in [x] \in A/L .$$

This is the regular left A-module, and its corresponding representation is the left regular representation on A/L with kernel

$$\{a \in A : (aA)' = \{0\}\} = \{a : aA \subseteq L\}$$

2.6 Definition A left A-module is non-trivial if $AX \neq \{0\}$. An irreducible left A-module is a non-trivial left A-module such that

X and $\{0\}$ are the only A-submodules of X, and a representation of A is irreducible if the corresponding left A-module is irreducible.

2.7 Definition If x_o is in a left A-module X, we denote

$$\ker(x_o) = \{a \in A : ax_o = 0\}$$

and call x_o cyclic if $Ax_o = X$.

2.8 Theorem [4:120] If X is an irreducible left A-module and $x_o \in X \backslash \{0\}$ then x_o is a cyclic vector and $\ker(x_o)$ is a maximal modular left ideal.

2.9 Theorem [4:120] If J is a maximal modular left ideal of A, then A/J is irreducible.

If X is an irreducible left A-module, we consider a special subset of L(X): $\mathcal{D} = \{T \in L(X) : aTx = T(ax)\, a \in A, x \in X\}$.

2.10. Theorem [24:35] $\mathcal{D} = \underline{C}I$, where I is the identity operator.

2.11 Definition Vectors $x_1, \ldots x_n$ in an irreducible left A-module X are $\mathcal{D}$-independent if $D_1, \ldots, D_n \in \mathcal{D}$ and

$$D_1x_1 + \ldots + D_nx_n = 0 \text{ implies } D_1 = D_2 = \ldots = D_n = 0.$$

2.12 Theorem [4:122] Let $x_1, \ldots, x_n$ be $\mathcal{D}$-independent vectors in an irreducible left A-module X. Then there exists an a in A such that

$$ax_k = 0 \ (1 \leq k \leq n-1) \text{ and } ax_n \neq 0.$$

Theorem 2.12 will play an important role in our next continuity result, but a few more definitions relating to ideals are necessary.

2.13 Definition If L is a left ideal of A, the quotient of L is the bi-ideal $L:A$ given by $L:A = \{a \in A : aA \subset L\}$.

The quotient of a maximal modular ideal is a primitive ideal.

2.14 Theorem [4:123] 1) The primitive ideals of A are the kernels of the irreducible representations of A.

2) A primitive ideal is the intersection of the maximal modular left ideals containing it.

2.15 Definition The (Jacobsen) radical of A is the intersection of the kernels of all representations of A. A is semisimple if $\mathrm{rad}(A) = 0$ and a radical algebra if $\mathrm{rad}(A) = A$.

In a commutative Banach algebra, Definition 2.15 and the definition of radical used in Chapter I are equivalent. It will probably be a relief to know that the next lemma will lead to the promised automatic continuity theorem.

2.16 Lemma [4:128] Let X be an irreducible Banach left A-module. Let A_o denote the closed unit ball in A and let $x_o \in X \backslash \{0\}$ If L is a closed left ideal of A with $L \not\subset \ker(x_o)$, then there exists $K > 0$ such that $KA_o x_o \subset (L \cap A_o) x_o$.

Proof Let $M = \ker(x_o)$ and a' denote the M-coset of a. Since by Theorem 2.8 M is a maximal modular left ideal and $L \not\subset M$, $L \oplus M = A$.

Therefore $a \to a'$ maps the Banach space L onto the Banach space A/M. By the Open Mapping Theorem, there exists $K > 0$ such that for every $y \in A/M$ with $\| y \| \leq K$, there exists $b \in L \cap A_0$ with $b' = y$. Given $a \in K A_0$, we have $\| a' \| \leq K$ and so there exists $b \in L \cap A_0$ with $b' = a'$. Thus $b - a \in \ker(x_0)$ and $ax_0 = bx_0$.

2.17 <u>Theorem</u> [4:128] Let $BL(X)$ denote the space of bounded linear mappings of X into itself. If π is an irreducible representation of [illegible] a normed linear space X such that $\pi(a) \in BL(X) (a \in A)$, then π [illegible]tinuous.

<u>Proof</u> The proof can be reduced to the case where $\ker(\pi) = \{0\}$ as we now show. Let $K = \ker(\pi)$, so $K = \{a \in A : aX = \{0\}\} = \bigcap \{\ker(x) : x \in X \setminus \{0\}\}$.

Since $\ker(x)$ is a maximal modular ideal for each $x \in X \setminus \{0\}$, and since $\ker(\pi)$ is a primitive ideal (Theorem 2.14), K is a closed bi-ideal and $B = A/K$ is a Banach algebra. Define τ on B by

$$\tau[b]x = \pi(a)x \quad (a \in [b] \in B, x \in X).$$

$\tau[b]$ is a well defined linear operator on X and

$$\| \tau[b]x \| = \| \pi(a)x \| \leq \| \pi(a) \| \; \| x \| \quad , \text{ so } \tau[b] \in BL(X).$$

If τ is continuous, so is π, and τ is an irreducible representation of B on X with $\ker(\tau) = \{0\}$.

So assume $\ker(\pi) = 0$. If X is finite dimensional, so is $L(X)$, and since $\ker(\pi) = 0$, A has finite dimension. Thus π is continuous.

We therefore assume that X is infinite dimensional. Given $x \in X$, let $\sigma(x)$ be the linear mapping of A into X defined by

$$\sigma(x)a = ax \; (a \in A).$$

Let $Y = \{x \in X : \sigma(x) \in BL(A,X)\}$. Y is an A submodule, because for $y \in Y$ and $b \in A$ we have

$$\sigma(by)a = aby = \sigma(y)(ab) \text{ and}$$

$$\| \sigma(by)a \| \leq \| \sigma(y) \| \, \|ab\| \leq \| \sigma(y) \| \, \| a \| \, \| b \| .$$

Since X is irreducible, either $Y = X$ or $Y = \{0\}$. Suppose first that $Y = X$, and denote the closed unit ball by X_o. Then

$$\| \sigma(x)a \| = \| ax \| = \| \pi(a)x \| \leq \| \pi(a) \| \qquad (x \in X_o,\ a \in A).$$

By the Uniform Boundedness Theorem, there exists $m > 0$ such that

$\| \sigma(x) \| \leq m$. But this means $\| ax \| \leq m \| a \|$, or

$\| \pi(a) \| \leq m \| a \|$ and π is continuous.

Now suppose $Y = \{0\}$ and let A_o be the closed unit ball of A. By definition of Y, $A_o x$ is unbounded where $x \in X\setminus\{0\}$. Since X has infinite dimension, X contains an infinite sequence $\{x_n\}$ of $\mathcal{D}$-independent vectors, and we may take $\| x_n \| = 1$.

Let $M_n = \ker(x_n)$ and $L_n = M_1 \cap \ldots \cap M_{n-1}$. By Theorem 2.12, there is an $a \in A$ with $ax_K = 0$ $(1 \leq K \leq n-1)$ and $ax_n \neq 0$; in other words $a \in L_n \setminus M_n$. Therefore $L_n \not\subset M_n$ and since $A_o x_n$ is unbounded, Lemma 2.16 shows that $(L_n \cap A_o)x_n$ is unbounded.

Choose $a_1, \ldots, a_n \ldots$ with $a_n \in L_n$, $\| a_n \| < 2^{-n}$ and

$$\| a_n x_n \| > n + \| (a_1 + \ldots + a_{n-1})x_n \| .$$

Let $b = \sum_{n=1}^{\infty} a_k$ and $b_n = \sum_{k=n+1}^{\infty} a_k$. We have $a_k \in M_n (k > n)$ and since M_n is closed, it follows that $b_n \in M_n$. Therefore $b_n x_n = 0$. But $b = a_1 + \ldots + a_n + b_n$, and so

$$bx_n = a_1 x_n + \ldots + a_n x_n \text{ which implies}$$

$$\| bx_n \| \geq \| a_n x_n \| - \| (a_1+\ldots+a_{n-1})x_n \| > n.$$

This contradicts the fact that $\pi(b) \varepsilon BL(X)$, and we conclude that $Y \neq \{0\}$. Thus $Y = X$ and π is continuous.

Theorem 2.17 was proved by Johnson [10] as the major tool in his proof of Theorem 2.18.

§2 Uniqueness of Complete Algebra Norms

The uniqueness of the complete *algebra norm for* *commutative* semisimple Banach algebras is an easy consequence of Theorem 1.3, and was proved as early as 1948 [4:131]. The non-commutative case is a very different matter, and Johnson's proof did not appear until 1967. The proof given here is found in Bonsall and Duncan [4:130], with a trivial addition to make use of the *separating space from* Chapter I.

2.18 Theorem Let $(A, \|.\|_1)$ be a semisimple Banach algebra, and let $\|.\|_2$ be a second algebra norm with respect to which A is complete. Then $\|.\|_2$ is equivalent to $\|.\|_1$.

Proof Let M be a maximal modular left ideal of A, X = A/M, and let $\|.\|_1', \|.\|_2'$ denote the canonical norms on X derived from $\|.\|_1$ and $\|.\|_2$. Because M is closed, X is a Banach space with respect to each norm. Let π denote the left regular representation of A on X. By Theorem 2.9, π is an irreducible representation of $(A, \|.\|_1)$ on the normed linear space $(X, \|.\|_2')$, and since

$$\| \pi(a)[x] \|_2' = \| a[x] \|_2' \leq \| a \|_1 \| [x] \|_2' \quad a \varepsilon A, [x] \varepsilon X,$$

$\pi(a)\varepsilon BL(X, \|\cdot\|_2')$ for each $a\varepsilon A$. Therefore, by Theorem 2.17, π is continuous and there exists a positive constant K such that

$$\|\pi(a)[x]\|_2' \leq K\|a\|_1\|[x]\|_2' \quad a\varepsilon A, [x]\varepsilon X.$$

Let $a \to a'$ be the canonical mapping of A onto A/M, and let u be a right modular identity for M. Then for every a in A, $au-a\varepsilon M$, and so

$$\pi(a)u = (au)' = a'.$$

Therefore,

$$\|[x]\|_2' = \|a'\|_2' = \|\pi(a)u\|_2' \leq K\|a\|_1\|u\|_2', \; a\varepsilon[x]\varepsilon A/M$$

Since this holds for all $a\varepsilon[x]$, we have

$$\|[x]\|_2' \leq K\|x\|_1\|u\|_2'$$

and we conclude that $\|\cdot\|_1'$ and $\|\cdot\|_2'$ are equivalent on A/M.

Now look at the identity map $I:(A, \|\cdot\|_1) \to (A, \|\cdot\|_2)$.

Let $a\varepsilon\mathfrak{S}(I)$, the separating space of I. Then there exists a sequence $\{a_n\}$ in $(A, \|\cdot\|_1)$ with $\|a_n\|_1 \to 0$ and $\|a_n - a\|_2 \to 0$. Thus $\|a_n'\|_1' \to 0$ and $\|a_n' - a'\|_2' \to 0$. Since the canonical norms are equivalent on A/M, $\|a_n' - a'\|_1' \to 0$, which implies that $a' = 0$ and $a\varepsilon M$. Since this holds for every maximal modular left ideal, $a\varepsilon\mathrm{rad}(A) = \{0\}$ by semisimplicity. Thus $\mathfrak{S}(I) = \{0\}$ and I is continuous. $I: (A, \|\cdot\|_2) \to (A, \|\cdot\|_1)$ is continuous as well, so the norms are equivalent.

An improvement of Theorem 1.5 is immediate.

2.19 <u>Corollary</u> All involutions on a semisimple Banach algebra are continuous.

<u>Proof</u> If $*$ is an involution on A, let $\|x\|_1 = \|x^*\| . \|{*}\|_1$ is a complete algebra norm and is therefore equivalent to $\|.\|$, so $*$ is continuous.

The proof of Corollary 1.13 can now be given, as promised.

<u>Proof of Corollary 1.13</u> Let R be the radical of A. $*$ induces an involution on A/R such that $[x]^* = [x^*]$ because $R^* = R$ [18:55]. A/R is semisimple [4:124], so the involution is continuous. Thus, there exists a constant m^2 such that $\|[x^*]\| \le m\|[x]\|$ for all x in A, so

$$\nu(b^*b) = \nu[b^*b] \le \|[b^*b]\| \le \|[b^*]\| \cdot \|[b]\| \le m^2 \|[b]\|^2 \le m^2\|b\|^2.$$

Thus, $|f(a^*ba)| \le mf(a^*a)\|b\|$ by Theorem 1.12.

If A has an identity, then

$|f(b)| \le mf(e)\|b\|$ for all b in A and f is continuous.

The study of norms has assisted the study of positive linear functionals, now we show its relation to homomorphisms.

§3 <u>A Problem of Kaplansky</u>

The following definition is probably familiar.

2.20 <u>Definition</u> If A is a Banach algebra with identity e, a <u>multiplictive semi-norm</u> on A is a function $|.|$ on A to $[0,\infty)$ satisfying

i) $|x+y| \le |x| + |y| \quad x,y\in A$

ii) $|xy| \le |x||y| \quad x,y\in A$

iii) $|\alpha x| = |\alpha||x| \quad \alpha\in \mathbb{C}, x\in A$

iv) $|e| = 1$

If $|x| = 0$ implies $x = 0$, $|.|$ is a multiplicative norm.

2.21 Theorem [2:592] Let A and B be Banach algebras with identities e and e' respectively. If Θ is a homomorphism of A into B with $\Theta(e) = e'$, then the function $|x| = \|\Theta(x)\|$, $x \varepsilon A$ is a multiplicative semi-norm on A. If $|.|$ is a multiplicative semi-norm on A, then there exists a homomorphism Θ of A into a Banach albebra B such that $|x| = \|\Theta(x)\|$, $x \varepsilon A$.

Proof One way is obvious. Suppose $|.|$ is a semi-norm on A. $I = \{x: |x| = 0\}$ is an ideal in A, and $|.|$ is constant on cosets of A/I because $|x-y| \geq |x| - |y|$. Thus, A/I is a normed algebra under the norm $|[x]| = |x|$. Let Θ be the canonical homomorphism of A into the completion of A/I, and we have what we desire.

Let X be a compact Hausdorff space, and let $C(X,\underset{\sim}{C})$ denote the algebra of continuous, complex-valued functions on X. With pointwise operations and the supremum norm, $C(X,\underset{\sim}{C})$ is a semisimple, commutative Banach algebra, so all complete algebra norms are equivalent. More is known for this algebra, for Kaplansky proved in 1949 [14:407] that any norm on $C(X,\underset{\sim}{C})$, whether complete or not, is greater than or equal to the supremum norm. The next theorem says the same thing.

2.22 Theorem [24:58] Let X be compact Hausdorff. If Θ is a monomorphism from $C(X,\underset{\sim}{C})$ into a Banach algebra B, then $\|\Theta f\| \geq \| f \|$ for all $f \varepsilon C(X,\underset{\sim}{C})$.

Proof By restricting attention to $\overline{\theta(C(X,\underset{\sim}{C}))}$, we may assume B is commutative with identity. Recall [21:328] that X is homeomorphic to $\Delta_{C(X,\underset{\sim}{C})}$ and that Δ_B is compact in the Gelfand topology. Then Θ induces a map $\Theta^*: \Delta_B \to X$ defined by $f(\Theta^*\psi) = \psi\theta(f)$ for all ψ in Δ_B and f in $C(X,\underset{\sim}{C})$ [16:136]. Θ^* is continuous by definition of the Gelfand topology, so $\Theta^*(\Delta_B)$ is compact and therefore closed in X.

Suppose that Θ^* is not onto. Then there exists $\lambda_o \varepsilon X \backslash \Theta^*(\Delta_B)$, and there are disjoint open U, V with $\lambda_o \varepsilon U$ and $\Theta^*(\Delta_B) \subseteq V$. By Urysohn's Lemma, we choose $f \varepsilon C(X,\underset{\sim}{C})$ such that $f(X\backslash V) = 1$, $f(\Theta^*(\Delta_B)) = 0$ and $g \varepsilon C(X,\underset{\sim}{C})$ such that $g(\lambda_o) = 1$, $g(X\backslash U) = 0$. Thus, $fg = g$.

Since $f(\Theta^*(\Delta_B)) = 0$, it follows by definition of Θ^* that $\psi(\Theta f) = 0$ for all $\psi \varepsilon \Delta_B$, and so $\Theta f \varepsilon \mathrm{rad}(B)$. Therefore $(1 - \Theta f)$ is invertible [20:265]. Since $fg = g$, $\Theta f \Theta g = \Theta g$ and $(1 - \Theta f)\Theta g = 0$. But this implies that $\Theta g = 0$, which contradicts the fact that Θ is a monomorphism. Thus, $\Theta^*(\Delta_B) = X$.

Finally, if $f \varepsilon C(X,\underset{\sim}{C})$,

$$\|f\| = \sup_{x \varepsilon X} |f(x)| = \sup_{\psi \varepsilon \Delta_B} |f(\Theta^* \psi)| = \nu(\Theta f) \quad [20:268]$$

$$\leq \|\Theta f\| \quad \text{as required.}$$

Kaplansky's theorem naturally raised the question, "Does there exist an incomplete algebra norm on $C(X,\underset{\sim}{C})$?" From Theorem 2.22, this is equivalent to asking "Is there a discontinuous monomorphism from $C(X,\underset{\sim}{C})$?"

This question is the problem of Kaplansky, and its solution is the aim of the next two chapters.

CHAPTER III

TOWARD A SOLUTION

Kaplansky's problem did not immediately arouse much interest, and few papers on the subject appeared for about eleven years. In 1960, Badê and Curtis published a fundamental paper [2] which, along with providing a considerable simplification of the problem of Kaplansky, encouraged further work on the continuity of homomorphisms in general.

§1 Results of Badê and Curtis

The main tool used by Badê and Curtis is the following interesting boundedness theorem.

3.1 Theorem [2:592] Let A be a commutative Banach algebra and Θ a homomorphism of A into a Banach algebra B. If $\{g_n\}$ and $\{h_n\}$ are sequences from A satisfying

i) $g_n h_n = g_n \quad n = 1,2, \ldots$

ii) $h_m h_n = 0 \quad m \neq n$

then $\sup \| (g_n) \| / \|g_n\| \|h_n\| < \infty$

Proof Suppose for contradiction that $\limsup \| \Theta(g_n)\| / \|g_n\| \, \|h_n\| = +\infty$ We may suppose $\|g_n\| = 1$, $n = 1,2, \ldots$. By (i), $\|h_n\| \geq 1$. We shall construct a linear combination of h_n's which maps into an element of infinite norm. Select distinct elements q_{ij}, $i, j = 1,2 \ldots$

from the sequence g_n such that

$$(+) \quad \| \Theta(q_{ij}) \| \geq 4^{i+j} \| p_{ij} \|$$

where p_{ij} is the relative unit corresponding to $g_m = q_{ij}$. Define

$$f_i = \sum_{j=1}^{\infty} q_{ij}/2^j \quad (i = 1,2 \ldots).$$

$p_{ij} f_i = 2^{-j} q_{ij}$, and this together with (+) shows $\Theta(f_i) \neq 0$. For each integer i, select j(i) large enough so that $2^{j(i)} > \| \Theta(f_i) \|$ and define

$$y = \sum_{i=1}^{\infty} p_{ij(i)}/2^i \| p_{ij(i)} \|$$

This yields

$$f_i y = q_{ij(i)}/2^{i+j(i)} \| p_{ij(i)} \| \qquad i = 1,2 \ldots$$

By (+) and the way j(i) was selected we have

$$\| \Theta(y) \| \; \| \Theta(f_i) \| \geq \| \Theta(f_i y) \| \geq 2^{i+j(i)} > 2^i \| \Theta(f_i) \|$$

Thus, $\| \Theta(y) \| > 2^i$ for every i, and this contradiction establishes the result.

Virtually every paper extending the work of Badé and Curtis utilizes a theorem similar to Theorem 3.1, and for this reason we present a very short example of the kind of assistance it provides.

Let A be a commutative semisimple Banach algebra with identity. By means of the Gelfand map A may be treated as an algebra of continuous functions on Δ_A [20:268]. Also assume that A is regular, that is, for any disjoint closed sets E and F in Δ_A, there exists a function in A which is zero on E and one on F.

3.2 Definition Denote by G the family of all sets $E \subseteq \Delta_A$ such that

$$\sup \| \Theta(g) \| / \| g \| \, \| h \| = m < \infty$$

for all functions g and h with support in E and such that gh = g.

3.3 Lemma [2:595] If $\{E_n\}$ is any sequence of disjoint open sets in Δ_A, then $E_n \varepsilon G$ for all sufficiently large n.

Proof If the lemma were not true, there would exist an infinite sequence $\{E_m\}$ of disjoint open sets and functions g_m, h_m in A with support in E_m such that

i) $\| g_m \| = 1$

ii) $g_m h_m = g_m$ and

iii) $\| \Theta(g_m) \| > m \|h_m\|$

and this contradicts Theorem 3.1.

Through a series of lemmas involving repeated applications of Theorem 3.1, G is shown to contain a maximal open set whose complement is finite.

3.4 Theorem [2:597] Let A be a commutative, regular Banach algebra with identity, and let Θ be an arbitrary homomorphism into a Banach algebra B. Then there exists a finite set F (the singularity set of Θ) of points in Δ_A and a constant m such that

$$\| \Theta(g) \| \leq m \| g \| \, \| h \| \quad \text{for all functions g and h in A}$$

having support in $\Delta_A \setminus F$ and such that gh = g.

Restricting our attention once again to $C(X,\underset{\sim}{C})$ for X compact Hausdorff, let Θ be a homomorphism of $C(X,\underset{\sim}{C})$ into a Banach algebra B and let $F = \{\omega_1, \ldots, \omega_n\}$ be the singularity set of Θ.

3.5 <u>Definition</u> $M(f)$ = the intersection of the maximal ideals

$M(\omega_i) = \{f \varepsilon C(X,\underset{\sim}{C}) : f(\omega_i) = 0\}$.

$J(F)$ = the ideal of functions each of which vanishes in a neighbourhood of F, the neighbourhood depending on the function.

$A(F)$ = the dense subalgebra of $C(X,\underset{\sim}{C})$ consisting of those functions f such that $f(\omega) = f(\omega_i)$ in a neighbourhood of each point $\omega_i \varepsilon F$, the neighbourhood varying with f.

With these definitions, Theorem 3.4 may be strengthened.

3.6 <u>Theorem</u> [2:599] If Θ is a homomorphism of $C(X,\underset{\sim}{C})$ into a Banach algebra with singularity set F, then Θ is continuous on $A(F)$.

<u>Proof</u> By Theorem 3.4 and the fact that the h_n's may be chosen to have norm 1 [2:598], we have

$$\| \Theta(g) \| \leq M \| g \| \quad g \varepsilon J(F),\ M \text{ a positive constant.}$$

Now choose functions e_i, $0 \leq e_i \leq 1$ such that $e_i e_j = 0$ when $i \neq j$ and $e_i(\omega) = 1$ in a neighbourhood of $\omega_i \varepsilon F$. Then for any $f \varepsilon A(F)$, $f - \sum_{i=1}^{n} f(\omega_i) e_i \ \varepsilon J(F)$, so

$$\| \Theta(f) \| \leq \| \Theta(f - \sum_{i=1}^{n} f(\omega_i) e_i \| + \| \sum_{i=1}^{n} f(\omega_i) \Theta(e_i) \|$$

$$\leq M \| f - \sum_{i=1}^{n} f(\omega_i) e_i \| + \| f \| \sum_{i=1}^{n} \| \Theta(e_i) \|$$

$$\leq [(n+1)M + \sum_{i=1}^{n} \| \Theta(e_i) \|] \| f \| \quad f \varepsilon A(F)$$

and Θ is bounded.

Since Θ is continuous on a dense subalgebra of $C(X,\underset{\sim}{C})$, it has a unique continuous extension μ to $C(X,\underset{\sim}{C})$ which agrees with Θ on $A(F)$.

3.7 Definition $\quad \lambda(f) = \Theta(f) - \mu(f) \quad f\varepsilon C(X,\underset{\sim}{C})$.

μ is the continuous part of Θ; λ the singular part.

The next important result reduces the problem of finding a discontinuous homomorphism from $C(X,\underset{\sim}{C})$ to that of finding a non-trivial homomorphism from a maximal ideal of $C(X,\underset{\sim}{C})$ to a radical Banach algebra.

3.8 Theorem [2:599] Let Θ be a homomorphism of $C(X,\underset{\sim}{C})$ into a commutative Banach algebra B and let R denote the radical of $\overline{\Theta(C(X,\underset{\sim}{C}))}$. Let $F = \{\omega_1, \ldots, \omega_n\}$ be the singularity set of Θ, μ and λ the continuous and singular parts of Θ, and M a constant such that $\| \mu(f) \| \leq M \| f \|$ for $f\varepsilon C(X,\underset{\sim}{C})$. Then

a) The range of μ is closed in B and

$\overline{\Theta(C(X,\underset{\sim}{C}))} = \mu(C(X,\underset{\sim}{C})) \oplus R$, a normed direct sum.

b) $R = \overline{\lambda(C(X,\underset{\sim}{C}))}$

c) $R \cdot \mu(M(F)) = 0$ and the restriction of λ to $M(F)$ is a homomorphism.

d) There exist linear transformations λ_i, $i = 1, \ldots, n$ such that:

i) $\lambda = \sum_{i=1}^{n} \lambda_i$

ii) $R = R_1 \oplus R_2 \ \ldots \oplus R_n$ where $R_i = \overline{\lambda_i(C(X,\underset{\sim}{C})}$

iii) $R_i \cdot R_j = 0$ when $i \neq j$ and $R_i \cdot \mu\ (M(\omega_i)) = 0$.

iv) The restriction of λ_i to $M(\omega_i)$ is a homomorphism.

<u>Proof</u> a) Let $A = \{f\colon \mu(f) = 0\}$. Since μ is continuous, A is a closed ideal in $C(X,\underset{\sim}{C})$ and there exists a closed set $G \subseteq X$ such that

$$A = \{f\colon f(\omega) = 0;\ \omega\varepsilon G\}\ [21:330]$$

If $C(X,\underset{\sim}{C})/A$ is given the canonical norm, then $C(X,\underset{\sim}{C})$ is isometrically isomorphic with $C(G)$ [18:236] and so

$$\|\,[f]\,\| = \sup_{\omega\varepsilon G}|f(\omega)|.$$

Also, the semi-norm $|f| = \|\mu(f)\|$ is constant on the cosets $[f]$, and so $C(X,\underset{\sim}{C})/A$ may be normed by defining $|[f]| = |f|$. By Theorem 2.22,

$$(*)\ |[f]| \geq \|\,[f]\,\|,\ f\varepsilon C(X,\underset{\sim}{C}).$$

But for any $g\varepsilon[f]$, we have $|f| = |g| \leq M\|\,g\|$ because μ is continuous. Thus

$$(**)\ \ |[f]| \leq M\inf\{\|\,g\|\ :\ g\varepsilon[f]\} = M\|\,[f]\,\|.$$

(*) and (**) show the norms are equivalent on $C(X,\underset{\sim}{C})/A$.

To show μ has closed range, suppose $b_o\varepsilon B$ and $b_o = \lim\mu(f_n)$. Then

$$\|\,[f_m - f_n]\,\| \leq |f_m - f_n| = \|\,\mu(f_m - f_n)\,\| \to 0$$

There exists $f_o\varepsilon C(X,\underset{\sim}{C})$ such that $\|[f_o - f_n]\| \to 0$. Thus

$$\|\mu(f_o) - \mu(f_n\| \le M \|[f_o - f_n]\| \to 0 \quad \text{and } b_o = \mu(f_o).$$

Thus $\mu(C(X,\underset{\sim}{C}))$ is algebraically isomorphic with $C(X,\underset{\sim}{C})/A$ and $\mu(C(X,\underset{\sim}{C})) \cap R = \{0\}$.

Now we show that $\lambda = \Theta - \mu$ maps into R. If $\emptyset \varepsilon \Delta_{B_o}$ where $B_o = \overline{\Theta(C(X,\underset{\sim}{C}))}$, then the functionals $\emptyset_\Theta$ and $\emptyset_\mu$ defined by $\emptyset_\Theta(f) = \emptyset(\Theta(f))$ and $\emptyset_\mu(f) = \emptyset(\mu(f))$ are complex homomorphisms and therefore continuous by Theorem 1.2. Since they coincide on the dense subalgebra $A(F)$, $\emptyset_\Theta = \emptyset_\mu$. Thus $\emptyset(\lambda(f)) = 0$ for $f \varepsilon C(X,\underset{\sim}{C})$, $\emptyset \varepsilon \Delta_{B_o}$, and $\lambda(f) \varepsilon R$.

Thus $\Theta(C(X,\underset{\sim}{C})) \subseteq \mu(C(X,C)) \oplus R$. We must show that $\overline{\Theta(C(X,\underset{\sim}{C}))} = \mu(C(X,\underset{\sim}{C})) \oplus R$. If $b = \lim \Theta(f_n)$, then since $\mu(C(X,\underset{\sim}{C}))$ is closed in B,

$$\begin{aligned} \|\Theta(f_m - f_n)\| &\ge \nu_B(\Theta(f_m - f_n)) \quad \text{(recall } \nu(x) \text{ is the spectral radius of Definition 1.9)} \\ &= \nu_B(\mu(f_m - f_n)) \\ &= \nu_{\mu(C(X,\underset{\sim}{C}))}(\mu(f_m - f_n)) \\ &\ge M^{-1} \|\mu(f_m - f_n)\|. \end{aligned}$$

There therefore exists an $f_o \varepsilon C(X,\underset{\sim}{C})$ such that $\mu(f_o) = \lim \mu(f_n)$.

If $r = b - \mu(f_o)$, then $r = \lim(\lambda(f_n))$ and $r \varepsilon R$. This completes (a), and proves (b) as a bonus.

c) Since J(F) is dense in $M(F)$, it is enough to show that

$\mu(g)\lambda(f) = 0$, $g \varepsilon J(F)$, $f \varepsilon C(X,\underset{\sim}{C})$. Now $fg \varepsilon J(F)$, and Θ and μ agree on J(F). Therefore:

$$\begin{aligned} \mu(g)\lambda(f) &= \mu(g)\,[\Theta(f) - \mu(f)] \\ &= \Theta(g)\Theta(f) - \mu(g)\mu(f) \\ &= \Theta(gf) - \mu(gf) = 0 \end{aligned}$$

If $f, g \in M(F)$, we have

$$\lambda(fg) = \Theta(fg) - \mu(fg)$$

$$= [\mu(f) + \lambda(f)][\mu(g) + \lambda(g)] - \mu(fg)$$

$$= \mu(f)\mu(g) + \lambda(f)\lambda(g) - \mu(fg)$$

$$= \lambda(f)\lambda(g)$$

and $\lambda: M(F) \to R$ is a homomorphism.

d) Choose functions e_i, $i = 1, \ldots, n$ such that e_i is one in a neighbourhood of ω_i and $e_i e_j = 0$ for $i \neq j$. Define

$$\lambda_i(f) = \lambda(e_i f) \qquad f \in C(X,\underset{\sim}{C}).$$

If $f, g \in M(\omega_i)$, then $e_i f, e_i g \in M(F)$ and so

$$\lambda_i(f)\lambda_i(g) - \lambda_i(fg) = \lambda((e_i^2 - e_i)fg) = 0$$

because $(e_i^2 - e_i)fg \in J(F)$. Thus, $\lambda_i : M(\omega_i) \to R$ is a homomorphism.

Since $(1 - \sum_{i=1}^{n} e_i)f \in J(F)$ for all $f \in C(X,\underset{\sim}{C})$,

$$0 = \lambda(1 - \sum_{i=1}^{n} e_i)f = \lambda(f) - \sum_{i=1}^{n} \lambda_i f \text{ and}$$

$$\lambda = \sum_{i=1}^{n} \lambda_i . \quad R_i \cdot R_j = 0 \text{ is immediate.}$$

We have two more results to prove: d(ii) and $R_i \cdot \mu(M(\omega_i)) = 0$.

To finish these, note that

1) $0 = \Theta(e_i)\Theta(e_j f) = \mu(e_i)[\mu(e_j f) + \lambda(e_j f)]$

$$= \mu(e_i)\ \lambda(e_j f) \quad i \neq j,\ f \in C(X,\underset{\sim}{C})$$

2) $\lambda(e_i f) = [\mu(1 - \sum_{j=1}^{n} e_j) + \sum_{i=1}^{n} \mu(e_j)]\, \lambda(e_i f)$

$= \mu(e_i)\lambda(e_i f)$ by 1) and (c).

d(ii) follows easily.

If $g \varepsilon M(\omega_i)$,

$g = \sum_{j=1}^{n} e_j g + (1 - \sum_{j=1}^{n} e_j)g$, and the last term is in $J(F)$.

Therefore by 1), and the fact that $e_i g \varepsilon M(F)$, we have

$$\lambda(e_i f)\mu(g) = \lambda(e_i f) \sum_{j=1}^{n} \mu(e_j)\mu(g)$$

$$= \lambda(e_i f)\mu(e_i g) = 0.$$

This finally completes the proof.

The state of affairs may now be summarized.

3.9 <u>Theorem</u> [2:602] If $C(X,\underset{\sim}{C})$, X compact Hausdorff, has any of the following, it possesses every other.

1) an incomplete multiplicative norm,

2) a discontinuous multiplicative semi-norm,

3) a discontinuous isomorphism into a Banach algebra,

4) a discontinuous homomorphism into a Banach algebra,

5) a homomorphism λ into a radical Banach algebra R with adjoined identity, such that for some maximal ideal $M(\omega_o)$, $\lambda(M(\omega_o)) \subseteq R$ and $\lambda(J(\omega_o)) = 0$.

Proof $(1) \leftrightarrow (3)$ and $(2) \leftrightarrow (4)$ follow from Theorem 2.21. $(3) \rightarrow (4) \rightarrow (5)$ because any one of the λ_i's of Theorem 3.9 may be appropriated. Given (5), the norm $|x| = \|x\| + \|\lambda(x)\|$ defines a multiplicative norm on $M(\omega_0)$ which can be extended to $C(X,\mathbb{C})$, so $(5) \rightarrow (1)$.

Successors to Badé and Curtis fall into two categories: those who attempted to prove results similar to Theorems 3.6 and 3.8 in more general settings, and those who sought extensions applicable to the problem of Kaplansky.

§2 Generalizations

Cleveland [5] initiated the study of automatic continuity under less restrictive circumstances by considering homomorphisms from non-commutative Banach *-algebras.

3.10 Theorem [5:1104] If Θ is an isomorphism of a C*-algebra A, then there exists a constant M such that

$$\|x\| \leq M\|\Theta(x)\| \quad , x \in A.$$

The similarity of the theorem to Theorem 2.22 is noted, but the difference in technique in obtaining it is emphasized. Cleveland comments [5:1098] that Theorem 3.10 implies every multiplicative norm on A is complete if and only if every isomorphism is continuous, and shows [5:1105] that if there is a discontinuous homomorphism on A, there is a discontinuous isomorphism on A.

Using the same technique as Badé and Curtis, Cleveland obtained a slight extension of Theorem 3.4. Let A be a C*-algebra with identity, $\Theta: A \to B$ a homomorphism with $B = \overline{\Theta(A)}$. If A' is a commutative C*-subalgebra of A containing the identity, A' is isometrically isomorphic to $C(X,\underset{\sim}{C})$ for some compact Hausdorff X [18:232]. We have

3.11 <u>Theorem</u> [5:1108] There exists a finite set F in X and a constant M such that

$$\|\Theta(g^2a)\| \leq M\|g\| \|ga\|, \ a\varepsilon A, \ g\varepsilon A', \text{ support of } g \subseteq X\backslash F.$$

Along different lines, Barnes [3] investigated homomorphisms with restricted range. Note first that an algebra is <u>strongly</u> semisimple if the intersection of all maximal modular two-sided ideals is zero.

3.12 <u>Definition</u> An algebra A is a <u>modular annihilator algebra</u> if for every maximal modular left ideal M and every maximal modular right ideal N,

1) $R(M) \neq 0$, $R(A) = 0$ where $R(M) = \{x\varepsilon A: yx = 0 \text{ for all } y\varepsilon M\}$

2) $L(N) \neq 0$, $L(A) = 0$ $L(N) = \{x\varepsilon A: xy = 0 \text{ for all } y\varepsilon N\}$

3.13 <u>Theorem</u> [3:1036] Let A be a Banach algebra which satisfies the property

(*) Whenever I is a closed ideal of A such that A/I is finite dimensional, then $I^2 = I$.

If $\Theta: A \to B$ is a homomorphism and B is a strongly semisimple modular annihilator algebra, then Θ is continuous.

That Barnes' theorem is not totally removed from the problem we are considering follows from his comment [3:1036] that every C*-algebra satisfies condition (*).

In 1967, Johnson [11] was able to show that for certain special classes of algebras of operators on a Banach space, all homomorphisms are continuous. More interesting, though, are his studies of arbitrary homomorphisms of C*-algebras using the ideal

$$J = \{t : t\varepsilon A,\ \Theta(t)s = 0\ \forall s\varepsilon\mathfrak{S}(\Theta)\}$$

where A is a C*-algebra and Θ is a homomorphism into a Banach algebra B.

3.14 Theorem [12:81] $\bar{J}$ is of finite codimension in A.

Theorem 3.14 has been stated only to allow the proof of a corollary which shows once more the usefulness of the separating space.

3.15 Corollary [12:83] If A is a C*-algebra with identity e and no closed proper ideals of finite codimension, then Θ is continuous.

Proof By hypothesis and Theorem 3.14, $\bar{J} = A$ unless $\bar{J} = \{0\}$ and A is finite dimensional, which is a trivial case. Thus $J = A$ and $\Theta(e)$, which is the identity of $\Theta(A)$, and thus the identity of $\overline{\Theta(A)}$, annihilates $\mathfrak{S}(\Theta)$. Therefore $\mathfrak{S}(\Theta) = \{0\}$ and Θ is continuous.

Theorem 3.14 applies to an algebra without identity, but in this case it is not possible to obtain a result analogous to Corollary 3.15. However, Johnson claimed that a direct generalization of Thereom 3.6 to C*-algebras would be achieved if it were possible to show that $\Theta|J$ is continuous. Motivated by this idea, Stein [27] concentrated on von Neumann algebras, and was able to generalize Theorem 3.8 to this

special case.

Thus encouraged, Stein pursued Johnson's suggestion further, but with less success. Again, let A be a C*-algebra and Θ a homomorphism into a Banach algebra B.

3.16 <u>Definition</u>

$$I_L = \{x \in A: \sup_{\|z\| \leq 1} \|\Theta(xz)\| < \infty\}$$

$$I_R = \{x \in A: \sup_{\|z\| \leq 1} \|\Theta(xz)\| < \infty\}$$

$$I = I_R \cap I_L$$

Stein showed that I_L is contained in Johnson's $\mathcal{J}$, and proved the following result.

3.17 <u>Theorem</u> [28:437] Let $U \subseteq I$ be a closed, two-sided ideal. Then $\Theta|U$ is continuous.

Continuing in the same vein, Sinclair [22] considered an ideal slightly different from Johnson's , namely

$$\mathcal{J}' = \{a \in A: \Theta(a)s = s\Theta(a) = \{0\} \quad \forall s \in \mathfrak{S}(\Theta)\}$$

Again through techniques broadly similar to those of Badé and Curtis, Sinclair obtained a partial generalization of Theorem 3.8 for non-commutative C*-algebras. It is illuminating to note that he could not prove part (b) of that theorem, which in the commutative case relied upon an application of Theorem 1.2. Two corollaries of his theorem must be mentioned.

3.18 Corollary [22:451] Let A be a C*-algebra, B a Banach algebra, and Θ a homomorphism from A to B. If B is continuous on all C*-subalgebras of A that are generated by single hermitian elements, then Θ is continuous.

3.19 Corollary [22:451] If there is a discontinuous homomorphism from any C*-algebra, then there is a discontinuous homomorphism from C[0,1], the algebra of continuous complex valued functions on the closed interval [0,1].

§3 Extensions Tailored to Kaplansky's Problem

The desire to gain further information about $C(X,\underset{\sim}{C})$ for X compact Hausdorff led Sinclair to investigate $C_o(\underset{\sim}{R})$, the continuous functions on the reals which vanish at infinity.

> If there is a discontinuous homomorphism from a C*-algebra into a Banach algebra, then there is a discontinuous homomorphism from C[0,1] into a Banach algebra (Corollary 3.19), and hence there is a discontinous homomorphism ψ from $C_o([0,1]\setminus\{\lambda\})$ into a radical Banach algebra for some λ in [0,1] by Theorem 3.9. By restricting ψ to $C_o[0,\lambda)$ or $C_o(\lambda,1]$ we obtain a discontinuous homomorphism from one of these into a radical Banach algebra, and ψ annihilates functions with compact support [2:603]. Identifying $[0,\lambda)$ or $(\lambda,1]$ with $[0,\infty)$ homeomorphically, and sending functions with support in $(-\infty,1]$ to zero, we obtain a discontinuous homomorphism from $C_o(\underset{\sim}{R})$ into a radical Banach algebra. [23:165]

This study also yielded:

3.20 Theorem [23:172] Let X be a compact Hausdorff space and suppose there is a discontinuous homomorphism from $C(X,\underset{\sim}{C})$ onto a dense subalgebra of a Banach algebra B. Then there is a closed ideal M in B such that $\psi: C(X,\underset{\sim}{C}) \to B/M$, defined by $\psi(f) = \Theta(f) + M$, is a discontinuous homomorphism whose kernel is a prime ideal in $C(X,\underset{\sim}{C})$.

A new level of complexity in the efforts to extend Theorem 3.8 was reached in the 1976 paper of Johnson [13], which undertook a more detailed analysis of the finite singularity set F. The methods of Badé and Curtis were used to find a finite set E such that

$$E \subseteq \beta(X\backslash F)\backslash(X\backslash F)$$

where $\beta(X\backslash F)$ is the Stone-Čech compactification of $X\backslash F$. A theorem paralleling Theorem 3.8 applies to the set.

Two other results in Johnson's paper which are of a different nature require some definitions. Let A be a totally ordered set. Write $A < x$ to mean $a < x$ for every a in A; similarly $x < A$, $A < B$. A subset T of A is <u>cofinal</u> (coinitial) if for each $a \varepsilon A$ there exists $t \varepsilon T$ such that $t \geq a$ ($t \leq a$).

3.21 <u>Definition</u> $(A, \leq)$ is an <u>η_1 set</u> if for any countable subsets T_1 and T_2 of A with $T_1 < T_2$, there exists $a \varepsilon A$ such that $T_1 < a < T_2$. This implies that an η_1-set is one in which no countable set is either coinitial or cofinal.

3.22 <u>Definition</u> An ordered field is <u>real-closed</u> if it has no proper algebraic extension to an ordered field. Equivalently, F is real-closed if every positive element is a square.

3.23 <u>Theorem</u> [13:46] Let E and F be real-closed η_1-fields containing the real numbers R and of cardinality $\aleph_1$. Then there is an isomorphism u of E onto F with $u|R$ = identity.

This theorem allows a surprising conclusion.

3.24 Corollary [13:38] Assuming the continuum hypothesis, if there exists a discontinuous homomorphism from $C(X,\underset{\sim}{C})$ for any compact Hausdorff X, then there exists a discontinuous homomorphism for each infinite dimensional $C(X,\underset{\sim}{C})$.

Taking stock of the situation, we see that Theorem 3.8 provided the major simplification of the problem of Kaplansky, and the latest theorems of Sinclair and Johnson indicated that investigating special cases would be profitable. Although the continuum hypothesis is anathema to some, Johnson's use of it was the inspiration for the solution of Kaplansky's problem discussed in Chapter IV.

CHAPTER IV

A DISCONTINUOUS HOMOMORPHISM FROM $C(X,\mathbb{C})$

In 1977 it was announced [7] that, working independently and employing completely different methods, H.G.Dales and J.Esterle had constructed discontinuous homomorphisms from $C(X,\mathbb{C})$ for X compact Hausdorff. Both constructions depended upon the assumption of the continuum hypothesis.

Dales' paper is an excellent example of the synthesis of diverse fields of mathematics in the solution of a particular problem, and the purpose of this chapter is to present a highly condensed account of his complicated result.

§1 Definitions and the Basic Idea

The work of Johnson suggested that consideration of an appropriate compact Hausdorff space would be sufficient for the problem, and for reasons which will become clearer in Theorems 4.2 and 4.20, Dales opts for $\beta\mathbb{N}$, the Stone-Čech compactification of the integers.

Select any $p\in\beta\mathbb{N}\setminus\mathbb{N}$ and define the maximal ideal

$\mathcal{M}(p) = \{f\in C(\beta\mathbb{N},\mathbb{R}): f(p) = 0\}$ and the ideal

$J(p) = \{f\in C(\beta\mathbb{N},\mathbb{R}): f^{-1}(0)$ is a neighbourhood of p in $\beta\mathbb{N}\}$.

Finally, identify c_0, the real valued sequences which converge to zero, with the set $\{f\in C(\beta\mathbb{N},\mathbb{R}): f|(\beta\mathbb{N}\setminus\mathbb{N}) = 0\}$.

In keeping with Theorem 3.8, the construction will yield a real linear homomorphism from $\mathcal{M}(p)$ to a radical Banach algebra that has kernel $J(p)$ but does not vanish on c_o.

4.1 Definition $A = \mathcal{M}(p)/J(p)$ $A_1 = C(\beta\underset{\sim}{N},\underset{\sim}{R})/J(p)$

A_1 is the algebra A with identity adjoined.

We note that the space $C(\beta\underset{\sim}{N},\underset{\sim}{R})$ is partially ordered by

$$f \leq g \text{ if } f(x) \leq g(x) \text{ for all } x \text{ in } \beta\underset{\sim}{N}.$$

Dales shows that the quotient order in A is a total order and proves the following theorem.

4.2 Theorem [6: Prop 2.7]

i) A is an η_1-set.

ii) The quotient field of A is a real-closed η_1-field which, assuming the continuum hypothesis, has cardinality $\aleph_1$.

The fact that we are dealing with $\beta\underset{\sim}{N}$ is used in Theorem 4.2 because under the continuum hypothesis, $C(\beta\underset{\sim}{N},\underset{\sim}{R})$ has cardinality $\aleph_1$ [9:185].

More importantly, it is shown that divisibility can be expressed in terms of this order, and this is written:

b divides a if and only if $|a| \underset{\sim}{\leq} |b|$.

This makes it possible to begin the construction of the homomorphism by finding a map λ on a subset of A and considering an algebraic problem.

If $a = bc$ in A and $\lambda(a)$ and $\lambda(b)$ are already defined, is it possible to solve $\lambda(a) = \lambda(b)x$? In case the range of λ is a radical Banach algebra R in which divisibility and order are related in the same way as in A, and λ is isotonic (that is, $|a| \lesssim |b|$ in A $\to$ $|\lambda(a)| \lesssim |\lambda(b)|$ in R), the answer is yes. For if $a = bc$, $|a| \lesssim |b|$, $|\lambda(a)| \lesssim |\lambda(b)|$ and so $\lambda(b)$ divides $\lambda(a)$. Thus, we can set $\lambda(c) = \lambda(a)\,\lambda(b)^{-1}$.

This problem is complicated by the fact that each element a in $A^+ = \{a \in A : a > 0\}$ is infinitely divisible; that is, for any integer n there exists a b in A such that $b^n = a$. Therefore, $\lambda(a)$ must also be a non-zero infinitely divisible element in the range. We pause for some definitions.

Let $\omega(t)$ be a positive measurable function on $[0,\infty)$ such that

$$\omega(s+t) \leq \omega(s)\omega(t) \text{ and } \lim \omega(t)^{1/t} = 0.$$

4.3 <u>Definition</u> 1. $L^1(\omega)$ is the space of equivalence classes (under equality almost everywhere) of Lebesgue measurable complex valued functions on $[0,\infty)$ with norm

$$\|f\| = \int_0^\infty |f(t)|\omega(t)\,dt \text{ and convolution multiplication}$$

$$(f*g)(t) = \int_0^t f(t-s)g(s)ds.$$

2. $L^1(0,1)$ is the Banach space of equivalence classes (under equality almost everywhere) of Lebesgue integrable complex valued functions on $[0,1]$ with norm

$$\|f\| = \int_0^1 |f(t)|dt \text{ and convolution multiplication}$$

$$(f*g)(t) = \int_0^t f(t-s)g(s)\,ds \quad 0 \leq t \leq 1.$$

$L^1(\omega)$ is a commutative, radical Banach algebra without identity [4:8] and so is $L^1(0,1)$ [16:29].

4.4 <u>Definition</u> For $\sigma > 0$ let

$$L_\sigma = \{f : f \text{ complex valued, measurable on } [0,\infty),\ \int_0^\infty |f(t)|\, e^{-\sigma t}\, dt < \infty\}$$

equating functions equal almost everywhere.

$$\Lambda = \bigcup \{L_\sigma : \sigma > 0\}.$$

$L_\sigma(\underset{\sim}{R})$ and $\Lambda(\underset{\sim}{R})$ denote the real valued functions of L_σ and Λ.

With norm $\|f\|_\sigma = \int_0^\infty |f(t)|\, e^{-\sigma t}$ and convolution multiplication, L_σ is a commutative, semisimple Banach algebra without identity [4:8]. Λ is a linear associated commutative algebra. If $\sigma_1 < \sigma_2$,

$$L_{\sigma_1} \subseteq L_{\sigma_2} \subseteq \Lambda \subseteq L^1(\omega) \subseteq L^1(0,1).$$

$L^1(\omega)$ and $L^1(0,1)$ are two of the few radical Banach algebras in which non-trivial infinitely divisible elements exist, and are therefore candidates for the range of λ. Unfortunately, however, it is not always possible to tell exactly which elements are powers of other elements. This is where Definition 4.4 comes in, because it <u>is</u> possible to decide this question in Λ. Just how this is accomplished requires some more definitions.

4.5 <u>Definition</u> For $\sigma \geq 1$, let $\Omega_\sigma = \{z \in \underset{\sim}{C} : \operatorname{Re} z > 1, |z| > \sigma\}$

$A_\sigma = \{f \in C^*(\overline{\Omega}_\sigma, \underset{\sim}{C}) : f \text{ is analytic on } \Omega_\sigma\}$ where $C^*(\overline{\Omega}_\sigma, \underset{\sim}{C})$ is the space of bounded functions on $\overline{\Omega}_\sigma$.

A_∞ = direct limit of the A_σ's [15:219].

A_σ is an algebra with respect to pointwise operations, and we define $|F|_\sigma = \sup \{F(z) : z \in \overline{\Omega}_\sigma\}$ for $F \in A_\sigma$.

4.6 Definition For $\sigma \geq 1$, the set C_σ is the subset of A_σ consisting of the zero function together with the functions F which satisfy

1. $F(z)z^k$ is bounded for each integer k, $z \varepsilon \overline{\Omega}_\sigma$
2. $F(z) \neq 0 \ (z \varepsilon \overline{\Omega}_\sigma)$
3. $F(z) \varepsilon \mathbb{R} (z \varepsilon \overline{\Omega}_\sigma \cap \mathbb{R})$.

C_∞ = direct limit of the C_σ's, and is a subset of A_∞.

4.7 Definition Let $O(\Omega)$ be the algebra of analytic functions on the open subset Ω of $\mathbb{C}$. For $f \varepsilon L_\sigma$, the Laplace transform of f is

$$(\mathcal{L}f)(z) = \int_0^\infty f(t)e^{-zt}dt \quad (\text{Re}z > \sigma).$$

$\mathcal{L}f \varepsilon O(\Omega_\sigma)$ and $\mathcal{L}(f*g) = \mathcal{L}f\mathcal{L}g$ [25:171].

4.8 Definition If $F \varepsilon C_\sigma$, the inverse Laplace transform is

$$f(t) = (\mathcal{L}^{-1}F)(t) = \frac{1}{2\pi i} \lim_{y\to\infty} \int_{r-iy}^{r+iy} F(z)e^{zt}dz \quad t \geq 0,\ r \geq \sigma$$

and is independent of r for each $t \geq 0$ [25:175].

The question that generated the last four definitions can now be answered: the functions in Λ which are infinitely divisible are those whose Laplace transforms satisfy conditions (1) and (2) of Definition 4.6. We now have the barest essentials of Dales' construction, in that we desire an isotonic homomorphism from A into Λ. But to make things work properly, the space C_∞ must also be considered. Building a map from A to C_∞ occupies the bulk of Dales' paper.

§2 Triples and Extensions

Let A be an integral domain. If $\mathcal{B}$ is a subalgebra of A, then $\mathcal{B}$ is inverse closed if $ab_1 = b_2$ for $a \varepsilon A$, b_1, $b_2 \varepsilon \mathcal{B}$ with $b_1 \neq 0$ implies that $a \varepsilon \mathcal{B}$. If $\mathcal{B}$ is a subset of A, Alg $\mathcal{B}$ denotes the smallest inverse closed subalgebra of A that contains $\mathcal{B}$. We can now show how Dales relates A to C_∞

4.9 Definition $(\mathcal{Q}\,;\,\Theta;\mathcal{J})$ is a triple if

1. $\mathcal{Q}$ is an inverse closed subalgebra of A.
2. $\mathcal{J}$ is a closed subset of C_∞ which is a subalgebra of A_∞
3. $\Theta : \mathcal{Q} \rightarrow \mathcal{J}$ is an algebra isomorphism

A triple $(\mathcal{Q}_1;\,\Theta_1;\,\mathcal{J}_1)$ extends the triple $(\mathcal{Q}_2;\,\Theta_2;\,\mathcal{J}_2)$ if $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ and $\Theta_1|\mathcal{Q}_2 = \Theta_2$.

Dales notes the following result, the importance of which will soon be clear.

4.10 Theorem [6:Prop.4.17] If $\mathcal{Q}$ is an algebraically closed subalgebra of A, the quotient field of $\mathcal{Q}$ is a real-closed field.

The goal now is the exhibition of a partially ordered set of triples and to use a Zorn's lemma argument to deduce the existence of a maximal triple, providing an isomorphism from a maximal subset of A into C_∞. Naturally, this requires the existence of any triples at all, and this brings us to the foundation of the construction.

The first difficult portion of the paper involves producing such an 'initial' triple, and this is done through the careful (and lengthy) construction of what Dales christens a "framework map" from a certain subset of A to a subset of $\mathcal{C}_\infty$ [6: Chapters 2 and 3]. The subset of A selected contains elements of $c_0/J(p)$ [6:Def.2.9], and in keeping with the requirement that the nascent homomorphism be non-trivial, the range of the framework map is a subset of $\mathcal{C}_\infty$ that does not contain zero [6:Def.3.15]. The fact that A is an η_1-set is important both in choosing the subset of A and in building the framework map. This is where the algebraic problem of Section 1 is solved, because the map is isotonic. The framework map is obtained in [6:Thm.3.16] and is used to show the existence of a triple in [6:Prop.4.1].

If $(\mathcal{Q};\Theta;\mathcal{J})$ is a triple and $a\in A\backslash\mathcal{Q}$, the next problem is the extension of Θ to an isomorphism from Alg $(\mathcal{Q},a)$, the smallest inverse closed subalgebra of A containing $\mathcal{Q}$ and a. In the case that a is algebraic over $\mathcal{Q}$, one chapter is required to prove

4.11 <u>Theorem</u> [6:Thm 4.9] Let $(\mathcal{Q};\Theta;\mathcal{J})$ be a triple and let $\mathcal{Q}_1$ be the algebraic closure of $\mathcal{Q}$ in A. Then there is a triple $(\mathcal{Q}_1;\Theta;\mathcal{J}_1)$ extending $(\mathcal{Q};\Theta;\mathcal{J})$.

If a is transcendental over $\mathcal{Q}$, the problem is even more difficult, and Dales admits being unable to show the existence of an extension in general. However, if the functions in $\mathcal{J}$ are <u>approximable</u> (a property which requires several pages to define), an extension can be found [6:Thm 6.2].

Approximability is also important because the collection of triples $(\mathcal{Q};\Theta;\mathcal{J})$ with the property that each member of $\mathcal{J}$ is approximable may be partially ordered by $(\mathcal{Q}_1;\Theta;\mathcal{J}_1) \geq (\mathcal{Q}_2;\Theta;\mathcal{J}_2)$ if the first is an extension of the second. Every totally ordered subset has an upper bound, and so a maximal element exists by Zorn's lemma.

4.12 Definition $(\mathcal{Q}_*;\Theta;\mathcal{J}_*)$ is a triple which is a maximal member of the aforementioned partially ordered set.

4.13 Theorem [6:Thm 6.4] $\mathcal{Q}_*$ is an algebraically closed η_1-subset of A of cardinality $\aleph_1$.

Theorem 4.10 yields the next corollary immediately.

4.14 Corollary The quotient field of $\mathcal{Q}_*$ is a real-closed η_1-set of cardinality $\aleph_1$.

Finally, Theorem 4.2, Corrollary 4.14 and Theorem 3.23 allow Dales to prove the next theorem, which provides the most important component of the homomorphism we have been seeking all along.

4.15 Theorem The algebras A and $\mathcal{Q}_*$ are isomorphic.

Now that the maximal triple has been obtained, we describe some properties of the inverse Laplace transform.

4.16 Theorem [6:Prop.7.5] Let $F,G \in \mathcal{C}_\sigma$, $f = \mathcal{L}^{-1}F$ and let r be any number with $r > \sigma$. Then

i) f is continuous on $[0,\infty)$, $f(0) = 0$ and $f \in L_r$.

ii) $f \in L_r(\mathbb{R})$

iii) $\mathcal{L}^{-1}(FG) = \mathcal{L}^{-1}(F) * \mathcal{L}^{-1}(G)$ in L_r

iv) $\mathcal{L}^{-1} : \mathcal{J}_* \to \Lambda(\mathbb{R})$ is a real linear algebra monomorphism.

It will come as no surprise that Theorem 4.16 is a result of the way everything has been defined. Dales knew what he needed, and set things up accordingly.

This sketch has been very hasty, but it has brought to the forefront all we need to solve Kaplansky's problem.

§3 The Solution of the Problem of Kaplansky

We begin this section with a collection of definitions

4.17 Definitions Let $\pi : \mathcal{M}(p) \to A$ be the natural quotient map.

$i : A \to \mathcal{Q}_*$ be the isomorphism of Theorem 4.15

$\Theta : \mathcal{Q}_* \to \mathcal{J}_*$ be the isomorphism of the triple $(\mathcal{Q}_*;\Theta;\mathcal{J}_*)$,

$\mathcal{L}^{-1} : \mathcal{J}_* \to \Lambda(\mathbb{R})$ be the inverse Laplace transform.

It is impossible to resist the following theorem, which is a fine case of seeing only the tip of an iceberg.

4.18 Theorem [6:Thm 7.6] Assuming the continuum hypothesis, there is a real linear homomophism $\lambda : \mathcal{M}(p) \to \Lambda(\mathbb{R})$ such that $\ker\lambda = \mathcal{J}(p)$.

Proof Let $\lambda = \mathcal{L}^{-1} \circ \Theta \circ i \circ \pi$. (!)

The next theorem achieves the goal stated before Definition 4.1. Note that an element a in an algebra is nilpotent if $a^n = 0$ for some integer n.

4.19 Theorem [6:Thm. 7.7] Assuming the continuum hypothesis, there exists a real-linear homomorphism from $M(p)$ into a commutative radical Banach algebra R such that the homomorphism has kernel J(p). The homomorphism is discontinuous, and R may have either of the following properties:

i) R is an integral domain and has a bounded approximate identity.

ii) R has a dense set of nilpotents and a bounded approximate identity.

That the homomorphism is discontinuous follows because J(p) is dense in $M(p)$. Dales shows that $L^1(\omega)$ satisfies (i) and $L^1(0,1)$ satisfies (ii), and it has been noted that $\Lambda \subseteq L^1(\omega) \subseteq L^1(0,1)$.

At long last, we present the solution to Kaplansky's problem . The proof shows again that Dales' original selection of $\beta\mathbb{N}$ was expedient.

4.20 Theorem [6:Thm.7.8] Let X be an infinite compact Hausdorff space. Then, assuming the continuum hypothesis, there exists a discontinuous monomorphism from $C(X,\mathbb{C})$ into a Banach algebra.

Proof Let R be either of the commutative radical Banach algebras of Theorem 4.19, and let R_1 be the algebra with identity adjoined. Let λ be the real-linear homomorphism of Theorem 4.19. Extend λ first to a real linear homomorphism $\lambda: C(\beta\mathbb{N},\mathbb{R}) \to R_1$ and then to a complex linear homomorphism by

$$\lambda(f + ig) = \lambda(f) + i\lambda(g) \qquad f,\ g \in C(\beta\underset{\sim}{N},\underset{\sim}{C}).$$

Let Y be a countable discrete subspace of X, which exists by [9:5], and let τ: $\underset{\sim}{N} \to Y$ be a homeomorphism.

By properties of the Stone-Čech compactification, extend τ to

$$\tau : \beta\underset{\sim}{N} \to \overline{Y}.$$

If $f \in C(X,\underset{\sim}{C})$, then $f \circ \tau \in C(\beta\underset{\sim}{N},\underset{\sim}{C})$. Let A be the direct sum $C(X,\underset{\sim}{C}) \oplus R_1$. A is a commutative Banach algebra with respect to the coordinatewise algebraic operations and the norm

$$\|(f,r)\|_A = \sup\{|f|_X, \|r\|\} \quad (f,r) \in A, |\cdot|_X \text{ is the}$$

supremum norm.

If $f \in C(X,\underset{\sim}{C})$, let $\mu(f) = (f, \lambda(f \circ \tau)) \in A$. Then

$$\mu : C(X,\underset{\sim}{C}) \to A$$

is the desired discontinuous monomorphism.

To answer Kaplansky's twenty-eight year old question in its original form, the norm

$$\|f\| = \|\mu(f)\|_A \qquad f \in C(X,\underset{\sim}{C})$$

is incomplete and dominates the supremum norm.

§4 Final Remarks

In a private communication, A.M.Sinclair informed me that R.Solovay of the California Institute of Technology had also constructed a discontinuous homomorphism from $C(X,\underset{\sim}{C})$, but I have been unable to secure a copy of his work and do not know what technique he used.

However, since the result was mentioned along with those of Dales and Esterle without special comment, it is likely that it also requires the continuum hypothesis.

Whether or not a construction is possible without this axiom is an open question whose resolution will, with any luck, require less time than Kaplansky's original problem.

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